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B.Sc.-I, Title- Sphere (Analytical Geometry of 3D.)

(1) A sphere of constant radius r passes through the origin and cuts the axes in A, B, C. Find the locus of the foot of the perpendicular from O to the plane ABC.

Soln. Let the co-ordinates of the points A, B, C be respectively $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$.

Then the equation of the plane ABC is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. — (1)

Also, the equation of the sphere through the points O, A, B, C is $x^2+y^2+z^2-ax-bx-cz=0$.

So that the co-ordinates of its centre are $\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$ and its radius $=\sqrt{\frac{a^2+b^2+c^2}{4}}$.

But the radius is given to be $=r$. $\therefore a^2+b^2+c^2=4r^2$ — (2)

Let $P(\alpha,\beta,\gamma)$ be the foot of the perpendicular from the origin O to the plane (1).

Now the equation of the line through O(0,0,0) and perpendicular to the plane (1) is $\frac{x}{1/a}=\frac{y}{1/b}=\frac{z}{1/c}=r$ (say).

The co-ordinates of any point on this line are

$\left(\frac{r}{a},\frac{r}{b},\frac{r}{c}\right)$ — (3)

If it is the foot of the perpendicular from O to the plane (1), then its co-ordinates will satisfy the equation of the plane.

Hence $\frac{1}{a}\cdot\frac{r}{a}+\frac{1}{b}\cdot\frac{r}{b}+\frac{1}{c}\cdot\frac{r}{c}=1$.

$\Rightarrow r\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=1$, $\therefore r=\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$

Then from (3), $\alpha=\frac{r}{a}=\frac{\frac{1}{a}}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$, $\beta=\frac{r}{b}=\frac{\frac{1}{b}}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$

$\gamma=\frac{r}{c}=\frac{\frac{1}{c}}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$

$\therefore \alpha^2+\beta^2+\gamma^2=\frac{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}{(1/a^2+1/b^2+1/c^2)^2}=\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}=a\alpha=b\beta=c\gamma$.

$\therefore a = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha}, b = \frac{\alpha^2+\beta^2+\gamma^2}{\beta}, c = \frac{\alpha^2+\beta^2+\gamma^2}{\gamma}$.

Putting the values of a, b, c in (a), we get

$(\alpha^2+\beta^2+\gamma^2)^2\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right) = 4r^2$.

Hence the locus of the (α,β,γ) is

$(x^2+y^2+z^2)^2 \cdot (x^{-2}+y^{-2}+z^{-2}) = 4r^2$. //

(2) Find the equation of the sphere through the circle $x^2+y^2+z^2=a^2,\ z=0$

Soln: Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad (1)$$

Then the intersection of this sphere by the plane $z=0$ must be the given circle $x^2+y^2+z^2=a^2$.

Now putting $z=0$ in (1), we get

$$x^2+y^2+2ux+2vy+d=0 \quad (2)$$

which should be the form $x^2+y^2 \cancel{+z^2} - a^2 = 0$ — (3)

Hence comparing (2) and (3), we get

$$u=0,\ v=0,\ d=-a^2.$$

Therefore the required sphere is $x^2+y^2+z^2+2wz-a^2=0$ /

(3) Find the co-ordinates of the centre and the radius of the circle $x^2+y^2+z^2-2y-4z=11,\ x+2y+2z=15$.

Soln: The equation of the sphere can be written as

$$x^2+(y^2-2y)+(z^2-4z)=11.$$

i.e. $(x-0)^2+(y-1)^2+(z-2)^2 = 11+1+4 = 16$.

Hence the centre of the sphere is $(0,1,2)$ and its radius $=4$.

Now the length of the pendicular from the centre $(0,1,2)$ on the given plane $x+2y+2z=15$ is

$$\frac{1\cdot 0+2\cdot 1+2\cdot 2-15}{\sqrt{1^2+2^2+2^2}} = -\frac{9}{3} = 3 \text{ (in magnitude)}.$$

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$\therefore$ (radius of the circle)^2 = (radius of the sphere)^2 − (the length of the perpendicular from the centre on the plane)^2 $= 16 - 9 = 7$. $\therefore r = \sqrt{7}$.

Now to find out the co-ordinates of the centre we proceed as follows:

The equation of the line passing through the centre of the sphere i.e. the point $(0,1,2)$ and perpendicular to the given plane is

$$\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{2} = r \text{ (say)}$$

The coordinates of any point on this line are $(r, 1+2r, 2+2r)$. If this is the foot of the perpendicular (i.e. the centre of the circle) from the centre of the sphere to the plane, then its co-ordinates will satisfy the equation of the plane. Hence

$$r + 2(1+2r) + 2(2+2r) = 15.$$

i.e. $9r + 6 = 15$, $\therefore r = 1$.

Hence the co-ordinates of the centre of the circle are $(1, 3, 4)$. //

(4) obtain the equations of the circle lying on the sphere $x^2+y^2+z^2-2x+4y-6z+3=0$ and having its centre at $(2,3,-4)$.

Soln. The required equations are the equations of the given sphere and of a plane through $(2,3,-4)$ whose normal is the line joining $(2,3,-4)$ to the centre of the sphere.

Now the centre of the sphere is $(1,-2,3)$.

Hence the d.c. of the line joining the centre $(1,-2,3)$ and the point $(2,3,-4)$ are proportional to $(1,5,-7)$.

Hence the equation of the plane is ~~centre of the sphere~~

$x+5y-7z+K=0$

Since the plane passes through the point $(2,3,-4)$,

therefore $2+5\cdot 3-7(-4)+K=0$

$\Rightarrow 2+15+28+K=0$, $\therefore K=-45$.

$\therefore$ The equation of the plane is $x+5y-7z-45=0$.

Hence the equation of the circle is

$$x^2+y^2+z^2-2x+4y-6z+3=0,$$

$$x+5y-7z-45=0.$$

(5) Find the equation of the sphere which passes through $(1,-2,3)$ and through the points of intersection of the spheres

$$x^2+y^2+z^2-5x-5y-z+8=0,$$

$$x^2+y^2+z^2-3x-2y+z+18=0.$$

Soln: The equation of the any sphere passing through the points of intersection of the given sphere is

$$x^2+y^2+z^2-5x-5y-z+8+\lambda(x^2+y^2+z^2-3x-2y+z+18)=0.$$

This will pass through the point $(1,-2,3)$ if

$$1+4+9-5+10-3+8+\lambda(1+4+9-3+4+3+18)=0$$

$\Rightarrow 24+\lambda.36=0$, $\therefore \lambda=-\frac{24}{36}=-\frac{2}{3}$.

Putting the value of λ, the equation of the sphere becomes

$$3(x^2+y^2+z^2-5x-5y-z+8)-2(x^2+y^2+z^2-3x-2y+z+18)=0$$

$$\Rightarrow x^2+y^2+z^2-9x-11y-5z-12=0 \quad //$$