(2) Dr. Ramjee Prasad. (Mathematics), (Honours)
B.Sc-I, Title - The S.D. between ~~between~~ two lines.

(1) Two st. lines $\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1}$, $\frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}$ are cut by a third lines whose d.c. are λ, μ, ν. Show that d, the length intercepted on the third line, is given by

$$d\times\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ \lambda & \mu & \nu \end{vmatrix}=\begin{vmatrix} \alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}.$$

Deduce the length of s.d. between the first two lines.

Proof Let the two given lines be cut by the third line PQ in two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Then the d.c. of PQ are proportion to $x_1-x_2, y_1-y_2, z_1-z_2$.

$$\therefore \frac{\lambda}{x_1-x_2}=\frac{\mu}{y_1-y_2}=\frac{\nu}{z_1-z_2}=\frac{1}{K} \text{ (say)} \quad (1)$$

Now, $d^2=(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2$

$=(\lambda^2+\mu^2+\nu^2).K^2$ [from (i)]

$=K^2$, since $\mu^2+\lambda^2+\nu^2=1$, $\therefore d=K$.

Since $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ lie on the two given lines, therefore

$$\frac{x_1-\alpha_1}{l_1}=\frac{y_1-\beta_1}{m_1}=\frac{z_1-\gamma_1}{n_1}=r_1 \text{ and } \frac{x_2-\alpha_2}{l_2}=\frac{y_2-\beta_2}{m_2}=\frac{z_2-\gamma_2}{n_2}=r_2.$$

$\therefore x_1=\alpha_1+l_1r_1,\ y_1=\beta_1+m_1r_1,\ z_1=\gamma_1+n_1r_1.$

$x_2=\alpha_2+l_2r_2,\ y_2=\beta_2+m_2r_2,\ z_2=\gamma_2+n_2r_2$

$\therefore x_1-x_2=\alpha_1-\alpha_2+l_1r_1-l_2r_2, \ldots\ldots$ etc.

Hence from (1), $x_1-x_2=\lambda K \Rightarrow \alpha_1-\alpha_2+l_1r_1-l_2r_2=\lambda.d.$

$\Rightarrow l_1r_1-l_2r_2+(\alpha_1-\alpha_2-\lambda d)=0$ etc.

Eliminating r_1 and r_2 from these equations, we get the first result.

For the second part, we observed that if PQ be shortest distance between the given ~~the~~ lines, then it must be perpendicular to both.

Hence $\lambda l_1 + \mu m_1 + \nu n_1 = 0$

$\lambda l_2 + \mu m_2 + \nu n_2 = 0$

Solving, we get

$$\frac{\lambda}{m n_1 - m_1 n} = \frac{\mu}{n l_1 - n_1 l} = \frac{\nu}{l m_1 - l_1 m} = \frac{1}{\sqrt{\sum (m n_1 - m_1 n)^2}} \quad \text{--- (2)}$$

since $\lambda^2 + \mu^2 + \nu^2 = 1$.

Now, $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ \lambda & \mu & \nu \end{vmatrix} = \lambda(m n_1 - m_1 n) + \mu(l_1 n - l n_1) + \nu(l m_1 - l_1 m)$

$= \sqrt{\{\sum (m n_1 - m_1 n)^2\}}$, [from (2)]

Hence the second part is obtained.

(2) Find the length and equation of the shortest distance between $x - y + z = 0 = 2x - 3y + 4z$.

and $x + y + 2z - 3 = 0 = 2x + 3y + 3z - 4$.

Soln. By following the usual method, the eqn. of the st lines $\{x - y + z = 0 = 2x - 3y + 4z\}$ and $\{x + y + 2z - 3 = 0 = 2x + 3y + 3z - 4\}$ can be written respectively as

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{1} \quad \text{--- (1)} \quad \text{and} \quad \frac{x-5}{3} = \frac{y+2}{-1} = \frac{z-0}{-1} \quad \text{--- (2)}$$

The general equation of any plane passing through the line of intersection of the second pair of planes $x + y + 2z - 3 = 0 = 3y + 2x + 3z - 4$ is

$(x + y + 2z - 3) + \lambda(2x + 3y + 3z - 4) = 0$

$\Rightarrow x(1 + 2\lambda) + y(1 + 3\lambda) + z(2 + 3\lambda) - (3 + 4\lambda) = 0$ --- (3)

The plane (3) will be parallel to (1) if

$1(1 + 2\lambda) + 2(1 + 3\lambda) + 1(2 + 3\lambda) = 0 \Rightarrow \lambda = -\frac{5}{11}$.

Putting the value of $\lambda = -5/11$, the eqn. (3) becomes

$x - 4y + 7z - 13 = 0$ --- (4)

Now, any point on (1) is $(0, 0, 0)$.

Hence the required S.D. = the ⊥ from (0,0,0) on the plane (4)

$$\frac{-13}{\sqrt{1+4^2+7^2}} = \frac{13}{\sqrt{66}} \text{ (in magnitude)}.$$

The equation of the line of S.D. will be the line of intersection of the plane ⊥ (2), passing through (0,0,0) and plane ⊥ (1) passing through (5,−2,0) the point through which (2) passes. //

(3) Find the distance of A(1,−2,3) from the line PQ through P(2,−3,5) which makes equal angles with the axes.

Solution: Let the direction cosines of the line PQ be l, m, n. It is given that $l = m = n$.

But $l^2+m^2+n^2 = 1$.

$l = m = n = 1/\sqrt{3}$.

Therefore, PQ is a line which passes through the point P(2,−3,5) and which has direction cosines $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Draw AB perpendicular to PQ.

Now, PB = the projection of PA on PQ.

$$= (2-1)\frac{1}{\sqrt{3}} + (-3+2)\cdot\frac{1}{\sqrt{3}} + (5-3)\cdot\frac{1}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}$$

Also, $PA^2 = (2-1)^2+(-3+2)^2+(5-3)^2 = 1+1+4 = 6$.

Therefore from the rt. ∠ed △ABP,

$$AB^2 = AP^2 - PB^2 = 6 - \frac{4}{3} = \frac{14}{3},$$

$$\therefore AB = \sqrt{14/3}.$$

which is the required distance.

(4) Find the locus of the point which ~~moves~~ so that its distance from the line $x=y=z$ is twice its distance from the plane $x+y+z=1$.

Soln. Let $A(x_1, y_1, z_1)$ be the moving point and PQ is the line $x=y=z$. Let the coordinates of P be (0, 0, 0), therefore the direction cosines of the line PQ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Then PQ = projection of AP on PQ

$$= \frac{1}{\sqrt{3}}(x_1-0) + \frac{1}{\sqrt{3}}(y_1-0) + \frac{1}{\sqrt{3}}(z_1-0)$$

$$= \frac{x_1+y_1+z_1}{\sqrt{3}}, \text{ and } AP^2 = x_1^2+y_1^2+z_1^2.$$

If d is the distance of A from the ~~plane~~ line PQ, then

$$d^2 = x_1^2+y_1^2+z_1^2 - \frac{1}{3}(x_1+y_1+z_1)^2.$$

Now, if d_1 be the distance of A from the plane $x+y+z=1$.

then $d_1 = \frac{x_1+y_1+z_1-1}{\sqrt{3}} \Rightarrow d_1^2 = \frac{(x_1+y_1+z_1-1)^2}{3}$.

But it is given that $d = 2d_1$. $\therefore d^2 = 4d_1^2$.

$$\Rightarrow x_1^2+y_1^2+z_1^2 - \frac{1}{3}(x_1+y_1+z_1)^2 = 4 \cdot \frac{(x_1+y_1+z_1-1)^2}{3}$$

$$\Rightarrow 3x_1^2+3y_1^2+3z_1^2 - (x_1+y_1+z_1)^2 = 4(x_1+y_1+z_1)^2 - 8(x_1+y_1+z_1)+4$$

$$\Rightarrow 5(x_1+y_1+z_1)^2 - 8(x_1+y_1+z_1)+4 = 3x_1^2+3y_1^2+3z_1^2.$$

$$\Rightarrow 2(x_1^2+y_1^2+z_1^2) + 10(x_1y_1+y_1z_1+z_1x_1) - 8(x_1+y_1+z_1)+4 = 0$$

$$\Rightarrow x_1^2+y_1^2+z_1^2 + 5x_1y_1 + 5y_1z_1 + 5z_1x_1 - 4x_1 - 4y_1 - 4z_1 + 2 = 0$$

$\therefore$ Locus of the point $A(x_1, y_1, z_1)$ is

$$x^2+y^2+z^2+5xy+5yz+5zx-4x-4y-4z+2=0$$