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B.Sc. – III, Title – Ring Homomorphism.

(1) Let F be a field. Then ring of polonomial F(x) is F(x) is Euclidean domain.

Solution: Let F(x) is ring of polonomials over the field F.

Then (i) $\forall f(x), g(x) \in F(x) \Rightarrow f(x).g(x) = g(x).f(x)$.

(ii) $\forall f(x) \in F(x), \exists 1 = g(x) = F(x)$. such that $1.f(x) = f(x) = f(x).1$.

(iii) $f(x).g(x) = 0 \Rightarrow f(x) = 0$ or $g(x) = 0$.

which shows that $F[x]$ is an integral domain. In order to show that $F[x]$ is a Euclidean domains, we consider a mapping $d : (F[x] - \{0\}) \to W$ such that $d(f(x)) = \deg(f(x)) \in W$ for every $f(x) \in F[x] - \{0\}$.

(iv) For any $f(x) \neq 0$, $g(x) \neq 0 \in F[x]$,

$$d(f(x)g(x)) = \deg(f(x)g(x)).$$
$$= \deg(f(x)) + \deg(g(x)).$$
$$\geq \deg(f(x)) \quad [\because \deg(g(x)) \geq 0].$$

$d(f(x).g(x)) \geq d(f(x))$.

(v) For $f(x), g(x) \neq 0 \in F[x]$, by division algorithm, we have

$$f(x) = q(x).g(x) + r(x).$$

where $r(x) = 0$ or $d(r(x)) < \deg(f(x))$.

i.e. $f(x) = q(x).g(x) + r(x)$. where $r(x) = 0$ or $d(r(x)) < d(f(x))$.

Thus (i) – (v) $\Rightarrow F[x]$ is a Eclidean domain.

(2) Show that $D = \{a + b\sqrt{-2} : a, b \in Z\}$ is a Euclidean domain.

Solution: It can easily proved that D is a non-zero integral domain.

Consider a mapping $d : D - \{0\} \to W$ such that

$d(a + b\sqrt{-2}) = a^2 + 2b^2$.

(i) $d\{(a + b\sqrt{-2})(c + d\sqrt{-2})\} = d(ac - 2bd + (ad + bc)\sqrt{-2})$.

$$= (ac-2bd)^2 + 2(ad+bc)^2$$
$$= (a^2+2b^2)(c^2+2d^2)$$
$$\geq (a^2+2b^2) \quad [\because c^2+2d^2 \geq 0]$$

$\therefore d(a+b\sqrt{-2})(c+d\sqrt{-2}) \geq d(a+b\sqrt{2})$

(ii) Let $x = a+b\sqrt{-2}$, $y = c+d\sqrt{-2} \neq 0 \in D$. Then

$$\frac{x}{y} = p+\sqrt{-2}\,q = u.$$

If p, q are integers, then $u \in D$ and $x = uy$ i.e. $r=0$

If p, q are not integers then $u = p+\sqrt{-2}\,q \notin D$.

Consider integers p', q' such that

$|p-p'| \leq |q-q'| \leq \frac{1}{2}$ and $p'+\sqrt{-2}\,q' = u'$.

Now, $\frac{x}{y} = u \Rightarrow x = yu$ i.e. $x = yu' + yu - yu'$.

$$\Rightarrow x = u'y + (u-u')y.$$

$$\Rightarrow x = u'y + r,$$

Where, $r = (u-u')y$; i.e. $d(r) = ((u-u')y)$

$$= d((p-p') + \sqrt{-2}(q-q')y)$$
$$= ((p-p')^2 + 2(q-q')^2)(c^2+2d^2)$$

$\Rightarrow d(r) \leq \frac{3}{4} d(y)$ i.e. $d(r) \leq d(y)$

Thus $x = u'y + r$, $r=0$ or $d(r) < d(y)$.

this show that D is ~~euclidean~~ euclidean domain

(3) Let D be a Euclidean domain Prove the following:

(i) $d(a) = d(-a)$, $a \neq 0 \in D$

(ii) $d(a) = 0 \Rightarrow a$ is unit of D.

(iii) $d(ab) = d(a)$ if b is unit.

(iv) $d(ab) > d(a)$ if b is non-unit.

Proof: (i) $d(a) = d(-1)(-a) \geq d(-a)$. $\because d(xy) \geq d(x)$.
i.e. $d(a) \geq d(-a)$.
Now, $d(-a) = d((-1)(a))$
$d(-a) \geq d(a)$.
Hence $d(-a) = d(a)$.
(ii) Let a be non-unit element such that $d(a) = 0$.
Now, $d(a) = d(1.a) \geq d(1)$.
$\Rightarrow d(1) \leq d(a) = 0$
which contradicts the fact that $d(a) \geq 0$.
Hence, a is unit of D.
(iii) Let b is unit of D. Then b^{-1} exists.
Now $d(a) = (b^{-1}(ba)) \geq d(ba)$.
$\Rightarrow d(a) \geq d(ab)$ but $d(ab) \geq d(a)$.
Therefore, $d(a) = d(ab)$.
(iv) Let b is non-unit element of D. Then by division algorithm we have $a = q(ab) + r$; $r = 0$ or $d(r) < d(ab)$.
If $r = 0$ then $a = q(ab)$ i.e. $a(1 - qb) = 0$.
In view of def. of integral domain. $1 - qb = 0$
($\because a \neq 0$) i.e. $q = b^{-1}$. Hence, b is unit element of D.
This is contradiction to the fact that b is non-unity.
Hence, $r \neq 0$ is true.
$a = q(ab) + r \Rightarrow r = a + (1 - q.b)$
$\Rightarrow d(r) = d(a(1 - q.b))$
$\Rightarrow d(r) \geq d(a)$.
$\Rightarrow d(ab) > d(a)$ $\because d(ab) > d(r)$.
(Proved)

Theorem (4) Prove that every field is an integral domain.

Proof: Let F be a field. Then it is necessary an integral domain. Consider a mapping $d:(F-\{0\})\to W$ such that $d(x)=1\ \forall x\neq 0\in F$.

Now, (i) For $x,y\neq 0\in F$; $d(xy)=1 \Rightarrow d(xy)=d(x)$.

(ii) For any $x,y\neq 0\in F$; $x=(xy^{-1})y \Rightarrow x=(xy^{-1})y+0$

$\Rightarrow r=0$.

Hence, (i) and (ii) $\Rightarrow$ F is a Euclidean domain.

Theorem (5) Let D be a Euclidean domain and let S an ideal of D. Then there exists an element $\alpha\in S$ such that $S=\{\alpha x : x\in D\}$.

Proof: Suppose $S=\{0\}$. Then $S=\{0x : x\in D\}$. If $S\neq\{0\}$ then there exists atleast one non-zero element $\alpha\in D$. With least $d(\alpha)\in W$. Let $\beta\in S$. Then $\exists\, q,r\in D$ such that $\beta=q\gamma+r$ where $r=0$ or $d(r)<d(\alpha)$.

If $r\neq 0$ then $d(r)<d(\alpha)$ which contradicts the fact $d(\alpha)$ is least value. Hence $r=0$ is true.

i.e. $\beta=q\gamma$; $q\in D$.

In general $S=\{\alpha x : x\in D\}$

Theorem Any Euclidean domain has a unity element.

Proof: Let D be a Euclidean domain. Then D is an ideal of ring D itself. Therefore, there is some $\alpha\in D$ such that

$$D=\{\alpha x : x\in D\}$$

As $\alpha\in D \Rightarrow \exists\, e\in D$ such that $\alpha e=\alpha$.

Suppose $y\in D$. Then $y=\alpha x : \alpha\in D$.

$\Rightarrow ye=(\alpha x)e$

$\Rightarrow ye=\alpha(xe)=\alpha(ex)$. $\because \alpha e=\alpha$

$=(\alpha e)x=\alpha x$. $\because y=\alpha x$.

$ye=y$.

$\Rightarrow e$ is unit element of D.