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B.Sc.-III, Title - Ring Theory

Theorem: Let R be a ring. Then

(i) $x0 = 0 = 0x \ \forall x \in R$.

(ii) $(-x).y = -xy = x(-y) \ \forall x, y \in R$.

(iii) $x(y-z) = xy - x.z, \ \forall x, y, z \in R$.

(iv) $(-x)(-y) = x.y \ \forall x, y, z \in R$.

(v) If $R \neq 0$ is having unity then $1 \neq 0$.

Proof (i) $x0 = x(0+0) = x0 + x.0$ ∵ Distributive laws.

$\Rightarrow x0 + (-x)0 = (x.0 + x0) + (-x0)$.

$\Rightarrow \qquad = x0 + (x0 + (-x0))$. (Associative law)

$\Rightarrow 0 = x.0$

Similarly, $0x = x$. Hence, $0x = 0 = x0$.

(ii) $(-x)y + xy = (-x+x)y$. ∵ Distributive laws.

$= 0y = 0$ [From(i)].

$\Rightarrow (-x)y = -xy$.

Similarly, $x(-y) = -xy$.

(iii) $x.(y-z) = x(y + (-z))$

$= xy + x(-z)$.

$= xy - xz$ [From(ii)].

(iv) $(-x)(-y) = -(-x)y$

$= -(-xy) = xy$.

(v) $\forall x \in R \Rightarrow x.1 = x$ and $x.0 = 0$.

If possible let $1 = 0$, then $x.1 = x.0 \Rightarrow x = 0$

But $R \neq (0)$, therefore, $1 \neq 0$.

(2) The set R of all 2×2 matrices of the form $\begin{bmatrix} a+ib & c+id \\ -c+id & a-id \end{bmatrix}$ or $\begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}$ forms a division ring under matrices addition and multiplication.

Solution: We have

$$R = M_{2\times 2} = \left\{ A = \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}; z_1, z_2 \in C \right\}$$

It is clear that R or $M_{2\times2}$ form a ring under matrix addition and multiplication. Now we shall chek other conditions.

(3) A ring R is without zero divisors iff both right and left cancellation laws holds in R.

Proof:- Suppose ring R holds both left and right cancellation laws. If possible let R is with zero divisors i.e. $ab = 0$; $a \neq 0$, $b \neq 0$

Then $ab = a0 \Rightarrow b = 0$ (by cancellation laws) which is a contradiction. Hence, R is without zero divisors.

Conversally: Suppose R has no divisors of zero. Let $a, b, c \in R$ such that $ab = ac$; $a \neq 0 \Rightarrow ab - ac = 0$

$$\Rightarrow a(b-c) = 0 \Rightarrow b = c \quad \because a \neq 0.$$

Hence, $ab = ac \Rightarrow b = c$.

Similarly, right cancellation law also holds in R.

(4) Let $\phi: R \to S$ be a ring homomorphism. Prove that

(i) $\phi(0) = 0'$; 0 and $0'$ are zeros of R and S respectively.

(ii) $\phi(-x) = -\phi(x)$, $\forall x \in R$.

(iii) If $1 \in R$ then $\phi(1)$ is unity of S.

(iv) If R is commutative ring and ϕ is onto then $\phi(R)$ is commutative ring.

Proof: (i) Let $x \in R$. Then $\phi(x) \in S$. Suppose 0 be zero of R. Then $\phi(x+0) = \phi(x)$.

$\Rightarrow \phi(x) + \phi(0) = \phi(x)$. $\quad \because \phi$ is homomorphism.

$\Rightarrow \phi(0)$ is zero of S. i.e. $\phi(0) = 0'$.

(ii) $\phi(x) + \phi(-x) = \phi(x-x)$

$\Rightarrow \phi(x) + \phi(-x) = \phi(x-x)$.

$\Rightarrow \phi(-x) = -\phi(x)$.

(iii) For any $x \in R$, $x \cdot 1 = x$.

$\Rightarrow \phi(x \cdot 1) = \phi(x)$.

$\phi(x) \cdot \phi(1) = \phi(x)$.

$\Rightarrow \phi(1)$ is unity of S.

(iv) For any $x, y \in R$,

$x \cdot y = y \cdot x$

$\Rightarrow \phi(x\, y) = \phi(y \cdot x)$.

$\Rightarrow \phi(x) \cdot \phi(y) = \phi(y) \cdot \phi(x)$. //

(5) Let $\phi: R \to S$ be a ring homomorphism. Then $\phi(R)$ is a subring of S.

Proof: Let α, β be any element of $\phi(R)$. Then by definition,

$\alpha = \phi(x)$, $\beta = \phi(y)$; $x, y \in R$.

Now, $\alpha - \beta = \phi(x) - \phi(y) = \phi(x-y)$

$\Rightarrow \alpha - \beta = \phi(x-y) \in \phi(R)$. as $x-y \in R$.

and $\alpha\beta = \phi(x) \cdot \phi(y) = \phi(xy)$

$\Rightarrow \alpha \cdot \beta = \phi(xy) \in \phi(R)$ as $x, y \in R$.

$\therefore \phi(R)$ is subring of S.

(6) Let $\phi: R \to S$ is a ring homomorphism. Then kernel K_ϕ of homomorphism is a two sided ideal of R.

Proof: Let K_ϕ be kernel of ring homomorphism $\phi: R \to S$. Then $K_\phi = \{x \in R : \phi(x) = 0 \in S\}$ and it is clear that K_ϕ is non-subset of S because $0 \in R \Rightarrow \phi(0) = 0 \in S$.

~~Now $x, y \in K_\phi \Rightarrow \phi(x) = 0'$, $\phi(y) = 0'$~~

$\Rightarrow \phi(x) - \phi(y) = 0' - 0'$

$\Rightarrow \phi(x-y) = 0'$

$\Rightarrow x-y \in K_\phi$. $\because x, y \in R \Rightarrow x-y \in R$

Again $\phi(rx) = \phi(r)\phi(x) = \phi(r) \cdot 0' = 0'$.

and $\phi(x, r) = \phi(x) \cdot \phi(r) = 0' \cdot \phi(r) = 0'$.

Hence, rx and $xr \in K_\phi$.

Thus, K_ϕ is a two sided ideal of R.

(7) Every residue class ring of R is a homomorphic image of R.

Proof:- Let V is any ideal (two sided) of R. Therefore, R/V is residue class ring of R. To show that

$R/V = R$ or $R = R/V$.

Define $f: R \to R/V$ such that

$f(x) = V + x \in R/V \quad \forall x \in R$.

$f(x+y) = V + (x+y) = (V+x) + (V+y)$.

$f(x+y) = f(x) + f(y)$.

and $f(xy) = V + xy$

$= (V+x)(V+y)$.

$f(xy) = f(x) \cdot f(y)$.

Thus, f is a homomorphism

i.e. $R/V = R$. (Proved).